EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 4, 2019, 1350-1359
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Monotone flows with dense periodic orbits

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Abstract. The main result is Theorem 1: A flow on a connected open set $X \subset \mathbb{R}^d$ is globally periodic provided (i) periodic points are dense in X, and (ii) at all positive times the flow preserves the partial order defined by a closed convex cone that has nonempty interior and contains no straight line. The proof uses the analog for homeomorphisms due to B. Lemmens *et al.* [27], a classical theorem of D. Montgomery [31, 32], and a sufficient condition for the nonstationary periodic points in a closed order interval to have rationally related periods (Theorem 2).

2010 Mathematics Subject Classifications: 37C65, 37C25, 57Sxx

Key Words and Phrases: Monotone dynamical systems, Periodic points, Topological transformation groups

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1. Introduction

Many dynamical systems (φ, X), especially those used as models in applied fields, are *monotone*: The state space X has a nontrivial (partial) order which the dynamic $\varphi := \{\varphi^t\}_{t\in\mathbb{R}}$ preserves in positive time:

$$y \geq x \implies \varphi^t y \geq \varphi^t x, \quad (t \geq 0).$$

The great virtue of monotone systems is that long-term behavior of trajectories tends to be comparatively simple. While there can be exotic invariant sets, it is commonly the case that there are large sets of initial states $x(0)$ for which $x(t)$ approaches the fixed point set as t becomes infinite. This holds for cooperative systems (1) when the inequalities (2) are strict (Hirsch [16]).

Example. Consider a population divided into n groups labeled $i = 1, \ldots, n$. At time t the state of the system is characterized by a vector $x(t) \in \mathbb{R}^n$ whose i'th component $x_i(t)$ is the size (or density, concentration, probability, etc.) of group i. The growth rate of the groups is governed by a system of differential equations in the positive orthant $\mathbb{R}^n_+ = [0, \infty)^n$, having Kolmogorov form ([23, 39]):

$$\frac{dx_i}{dt} = x_i \sum_{j=1}^{n} g_{ij}(x_1, \ldots, x_n), \quad x_i \geq 0 \qquad (i = 1, \ldots, n). \tag{1}$$

DOI: https://doi.org/10.29020/nybg.ejpam.v12i4.3534

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The system is *cooperative* (or "mutualist") if the growth rate of population i tends to increase with the size of each population $j \neq i$, modeled by

$$\frac{\partial g_{ij}}{\partial x_j} \geq 0, \qquad i \neq j. \tag{2}$$

When the functions g_{ij} are continuously differentiable, this assumption makes the positive-time solution process preserve the vector order on $\mathbb{R}^n$ determined by the cone $\mathbb{R}^n_+ := [0, \infty)^n$:

$$s \leq t \implies x_i(s) \leq x_i(t), \qquad (i = 1, \ldots, n).$$

If the inequality on partial derivatives in Equation (2) is reversed, the system is called *competitive*.

Another common dynamical property is *dense periodicity*: the set of periodic points is dense. Often considered typical of chaotic dynamics, this condition is closely connected to many other important dynamical topics: structural stability, ergodic theory, Hamiltonian mechanics, smoothness and so forth. But in contrast to monotonicity, dense periodicity is usually demonstrated only in certain compact sets.

The goal of this article is to show that flows that are both monotonic and densely periodic are rare, because they are *globally periodic*: Our main result, Theorem 1, implies that such a flow factors through an action of the circle group.

For a sampling of the large literature on monotone dynamics, consult the following works and references therein: [1–4, 6–9, 12, 15, 20, 25, 28–30, 34, 36, 38, 40, 44–46]. Surveys of order-preserving dynamical systems are given in [19, 26, 41, 42].

1.1. Terminology

Let $\mathbb{Z}$ denote the integers, $\mathbb{N}$ the nonnegative integers, $\mathbb{N}_+$ the positive integers, $\mathbb{R}$ the reals, and $\mathbb{Q}_+$ the positive rationals. $\mathbb{R}^d$ is d-dimensional Euclidean space.

A subset S of a topological space Y is given the induced topology. When Y has been specified, the closure of S is denoted by $\overline{S}$.

Maps between topological spaces are always assumed continuous.

Let X be an *ordered space*: a topological space endowed with a partial order relation generally symbolized by $\succeq$, and denoted formally as $(X, \succeq)$. If $(X', \succeq')$ is also an ordered space, a map $T\colon X \to X'$ is *monotone* provided $x \succeq y \implies Tx \succeq' Ty$.

We write $x \preceq y$ as a synonym for $y \succeq x$. If $y \succeq x$ and $y \neq x$, we write $y \succ x$, $x \prec y$. For sets $A, B \subset X$, the notation $A \succeq B$ means $a \succeq b$ for all $a \in A, b \in B$; and similarly for $\succ$ and so forth.

The partial order is always assumed to be *closed* as a binary relation: The sets

$$\{(x, y) \in X \times X\colon\ x \succeq y\}, \quad \{(x, y) \in X \times X\colon\ x \preceq y\}$$

are closed in $X \times X$. Consequently:

- For all $p \in X$, the sets $\{x \in X\colon\ x \succeq p\}$ and $\{y \in X\colon\ y \preceq p\}$ are closed in X.
- If $\lim_i a_i = a$, $\lim_i b_i = b$, and $a_i \succeq b_i$, then $a \succeq b$.

The *order interval* $[a, b]$ is the closed set $\{x \in X\colon\ a \preceq x \preceq b\}$; its interior is the *open order interval* $[[a, b]]$. We write $a \ll b$ to indicate $[[a, b]] \neq \emptyset$.

Let $f\colon\ Y \to Z$ be a map. For $k \in \mathbb{N}$, the k'th iterate $f^k\colon\ y \to y_k$ is the map defined recursively by:

$$y_0 = y, \quad y_k = f(y_{k-1}) \textit{ if } y_{k-1} \in Y.$$

The *fixed point set* of f is

$$\mathcal{F}(f) := \{x\colon\ f(x) = x\}$$

and the *periodic set* is

$$\mathcal{P}(f) := \textstyle\bigcup_k \mathcal{F}(f^k).$$

A *flow* on X is an indexed family $\psi := \{\psi^t\}_{t\in\mathbb{R}}$ of homeomorphisms $\psi^t\colon\ X \approx X$ such that

$$\psi^r \circ \psi^s = \psi^{r+s}, \qquad (r, s \in \mathbb{R}), \tag{3}$$

and the *evaluation map*

$$\mathsf{ev}_\psi\colon\ \mathbb{R} \times X \to X, \qquad (t, x) \mapsto \psi^t x \tag{4}$$

is continuous. When X is ordered, ψ is *monotone* provided the maps $\psi^t, t \geq 0$ are monotone.

Example. A flow on $\mathbb{R}^n$ defined by a cooperative system of differential equations (1), (2) is monotone for the *vector order*

$$x \geq y \iff x_i \geq y_i, \quad (i = 1, \ldots, n).$$

A set $Y \subset X$ is *invariant* under ψ if $\psi^t Y = Y$ for all $t \in \mathbb{R}$, and $\psi\big|Y$ denotes the flow in Y whose evaluation map (4) is

$$\mathsf{ev}_{\psi|Y}\colon\ (t, y) \mapsto \psi^t y, \qquad (t, y) \in \mathbb{R} \times Y.$$

The *orbit* of x is

$$\mathcal{O}(x) := \{\psi^t x\colon\ t \in \mathbb{R}\},$$

and the orbit of a set $S \subset X$ is

$$\mathcal{O}(S) := \bigcup_{x\in S} \mathcal{O}(x).$$

Since orbits are invariant and the flow on an orbit is transitive, distinct orbits are disjoint.

The *periodic set* of ψ is

$$\mathcal{P} = \mathcal{P}(\psi) := \bigcup_{t>0} \mathcal{P}(\psi^t)$$

and the *equilibrium set* is

$$\mathcal{E} = \mathcal{E}(\psi) := \bigcap_{t\in\mathbb{R}} \mathcal{F}(\psi^t).$$

If $p \in \mathcal{P} \setminus \mathcal{E}$, its *period* is

$$\mathsf{per}\,(p) = \mathsf{per}\,(p, \psi) := \min\{t > 0\colon\ \psi^t p = p\} > 0,$$

and $O(p)$ is a *cycle*. If $\mathsf{per}(p) = r > 0$, the flow $\psi\big|O(p)$ is topologically conjugate to the flow on the topological circle $\mathbb{R}/r\mathbb{Z}$ covered by the translational flow on $\mathbb{R}$. Since the flows on cycles are transitive, every cycle is *unordered:* no two points are related by $\succ$.

The flow is called:

- *densely periodic* if $\mathcal{P}$ is dense in X,
- *pointwise periodic* if $\mathcal{P} = X$,
- *globally periodic* if ψ^t is the identity map of X for some $t > 0$,
- *monotone* if ψ^t is monotone for all $t \geq 0$.

This is the chief result:

Theorem 1 (Main). *Assume:*

(H1) *X is a connected open set in d-dimensional Euclidean space $\mathbb{R}^d$.*

(H2) *$K \subset \mathbb{R}^d$ is a closed cone with nonempty interior that is* convex *(contains the line segment joining any two of its points),* solid *(has nonempty interior), and* pointed *(contains no straight line).*

(H3) *The order on X is defined as: $x \geq y \iff x - y \in K$.*

(H4) *φ is a monotone flow on X.*

(H5) *φ is densely periodic.*

Then φ is globally periodic.

2. Resonant flows

Let ψ denote a monotone flow on an arbitrary ordered space X.

Definition. A set $S \subset X$ is *resonant* and ψ is *resonant in S*, provided:

$$a, b \in S \cap \mathcal{P} \setminus \mathcal{E} \implies \mathsf{per}\,(a)/\mathsf{per}\,(b) \in \mathbb{Q}_+.$$

It is easy to see that:

- *If S is resonant, so is its orbit and every subset.*
- *The intersection of resonant sets is resonant.*
- *The union of resonant sets is resonant if their intersection meets a cycle,*

Theorem 2 (Resonance Criterion). *Assume*

$$p, q \in \mathcal{P} \setminus \mathcal{E}, \quad [p, q] \subset X, \quad p \prec q, \quad O(p) \not\prec O(q). \tag{5}$$

Then $[p, q]$ is resonant.

Proof. Set

$$\mathsf{per}\,(p) = r > 0, \quad \mathsf{per}\,(q) = s > 0. \tag{6}$$

I claim:

$$r/s \text{ is rational.} \tag{7}$$

This is trivial if $r = s$. To fix ideas, assume assume $r < s$, the case $r > s$ being similar. Set $\xi := r/s$. For all $n, m \in \mathbb{Z}$:

$$p = (\psi^r)^n p = \psi^{nr} p, \qquad q = (\psi^s)^m q = \psi^{ms} q.$$

Monotonicity implies

$$p = \psi^{nr} p \prec \psi^{nr} q = \psi^{nr+ms} q = \psi^{(n\xi+m)s} q,$$

whence

$$p \prec \psi^{(n\xi+m)s} q, \qquad (n, m \in \mathbb{Z}). \tag{8}$$

Assume *per contra* that ξ is irrational. Then

$$\Lambda := \{(n\xi + m)s\colon\ n, m \in \mathbb{Z}\}.$$

is dense in $\mathbb{R}$,* whence

$$\Gamma := \{\Phi^t q\colon\ t \in \Lambda\}$$

is dense in $O(q)$. As the partial order relation is closed, (8) implies

$$p \preceq \Gamma \subset \overline{\Gamma} = O(q). \tag{9}$$

Invariance of cycles implies $O(p) \preceq O(q)$ by (9), monotonicity of ψ, and transitivity of $\psi|O(p)$. Therefore disjointness of $O(p)$ and $O(q)$ implies $O(p) \prec O(q)$. Since this contradicts the hypothesis, (7) is proved.

Next we prove:

$$u \in [p, q] \cap \mathcal{P} \setminus \mathcal{E} \implies \frac{\mathsf{per}\,(u)}{\mathsf{per}\,(q)} \in \mathbb{Q}_+. \tag{10}$$

This is trivial if $u \in \mathcal{E}$ or $u = q$, so we assume $u \notin \mathcal{E}$ and $p \preceq u \prec q$. Note that $O(u) \nprec O(q)$ because otherwise $O(p) \prec O(q)$, contrary to hypothesis. Therefore (10) follows from (7).

Resonance of $[u, v]$ now follows: If $u, v \in [p, q] \cap \mathcal{P} \setminus \mathcal{E}$, then applying the claim to both u and v gives:

$$\frac{\mathsf{per}\,(u)}{\mathsf{per}\,(v)} = \frac{\mathsf{per}\,(u)}{\mathsf{per}\,(q)} \cdot \frac{\mathsf{per}\,(q)}{\mathsf{per}\,(v)} \in \mathbb{Q}_+.$$

Proposition 1. *Let $p, q \in \mathcal{P}(\psi) \setminus \mathcal{E}(\psi)$ satisfy* (5). *If $\mathcal{P}(\psi)$ is dense in $[p, q]$, there exists $l > 0$ such that:*

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*Equivalently: The orbit of a rotation of the circle $\mathbf{S}^1$ through an irrational multiple of π is dense in $\mathbf{S}^1$. This result is ancient, going back to Nicole Oresme in the 14th century! See Grant [11], Kar [21]. A short proof based on the pigeon-hole principle is in Speyer [43]. Stronger density theorems are in Bohr [5], Kronecker [24], Weyl [47, 48].

(a) $\mathcal{P}(\psi) \cap [p, q] = \mathcal{P}(\psi^l) \cap [p, q]$,

(b) $\mathcal{P}(\psi^l)$ *is dense in* $[p, q]$,

(c) $\psi^l[p, q] = [p, q]$.

Proof. Let $\mathsf{per}\,(p) = r > 0$. Because $[p, q]$ is resonant (Theorem 2), if $z \in [p, q] \cap \mathcal{P}(\psi) \setminus \mathcal{E}(\psi)$ there exists $r \in \mathbb{N}_+$ such that $\psi^r z = z$.

Since the periods of all points in $[p, q] \cap \mathcal{P}(\psi) \setminus \mathcal{E}(\psi)$ are rational multiples of r, there exist $m, n \in \mathbb{N}_+$ such that:

$$(\psi^r)^m p = p, \quad (\psi^r)^n q = q, \text{ and } \mathcal{P}(\psi) \cap [p, q] = \mathcal{P}(\psi^r) \cap [p, q],$$

validating (a) and (b) for $l := mnr$. Monotonicity implies $\psi^l[p, q] \subset [p, q]$, so (c) follows from (b) and continuity of ψ^l.

2.1. Proof of Theorem 1

Recall the hypotheses, assumed henceforth:

(H1) *X is a connected open set in d-dimensional Euclidean space $\mathbb{R}^d$.*

(H2) *$K \subset \mathbb{R}^d$ is a closed convex cone that has nonempty interior and contains no straight line.*

(H3) *The partial order relation on X is determined by K:*

$$x \geq y \iff x - y \in K.$$

(H4) *φ is a monotone flow on X.*

(H5) *φ is densely periodic.*

The conclusion is: *φ is globally periodic.*

Definition. A homeomorphism $T\colon X \approx X$ is:

- *densely periodic* if $\mathcal{P}(T)$ is dense in X,
- *pointwise periodic* if $\mathcal{P}(T) = X$,
- *globally periodic* if T^k is the identity map of X for some $k \in \mathbb{N}_+$.

A crucial ingredient in the proof of Theorem 1 is the recently proved analog for monotone homeomorphisms:

Theorem 3 (B. Lemmens *et al.* [27])**.** *Assume* (H1), (H2), (H3). *Then a monotone homeomorphism $T\colon X \approx X$ is globally periodic provided it is densely periodic.*†

†Conjectured in M. Hirsch [18], and proved for polyhedral cones K.

We will also use an elegant result from the early days of transformation groups:

Theorem 4 (D. Montgomery [31, 32]). *A homeomorphism of a connected topological manifold is globally periodic provided it is pointwise periodic.*‡

Proposition 2. *Let $x \in X \setminus \mathcal{E}$ be arbitrary. There is an open neighborhood*

$$W_x \subset X \setminus \mathcal{E}$$

of x, and a real number $l := l_x > 0$, such that:

$$\varphi^l | W_x \text{ is a globally periodic homeomorphism of } W_x.$$

Proof. Every $x \in X \setminus \mathcal{E}$ has an open neighborhood $V_x \subset X \setminus \mathcal{E}$ that contains no orbit. If not, there is a sequence $\{x_k\}$ in converging in X to x such that

$$t \in \mathbb{R} \implies \lim_{k\to\infty} \|\varphi^t x_k - x_k\| = 0,$$

whence $t \in \mathbb{R} \implies \varphi^t x = x$. But this gives the contradiction $x \in \mathcal{E}$.

As $\mathcal{P}(\varphi)$ is dense, there exist periodic points p_x, q_x such that:

$$p_x \ll x \ll q_x, \quad [p_x, q_x] \subset V_x, \quad O(p_x) \not< O(q_x).$$

Define W_x to be the open order interval $[[p_x, q_x]]$. By Proposition 1 there exists $l > 0$ such that $\varphi^l | W_x$ is densely periodic. Therefore Theorem 3 implies $\varphi^l | W_x$ is globally periodic.

To finish the proof of Theorem 1, observe that φ is pointwise periodic by Proposition 2. Therefore Theorem 4 implies φ is globally periodic. ∎

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‡For analogs of Montgomery's Theorem in countable transformation groups, see Kaul [22], Roberts [37], Yang [50]. Pointwise periodic homeomorphisms on compact metric spaces are studied in Hall & Schweigert [13].

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